Necessary conditions for having wormholes in f(R) gravity

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For a generic $f(R)$ which admits a polynomial expansion of at least third order (i.e. $\frac{d^3f}{dR^3}\neq 0$) we find the near-throat wormhole solution. Necessary conditions for the existence of wormholes in such $f(R)$ theories are derived for both zero and non-zero matter sources. A particular choice of energy-momentum reveals that the wormhole geometry satisfies the weak energy condition (WEC). For a range of parameters even the strong energy condition (SEC) is shown to be satisfied.

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I. INTRODUCTION

Black hole solutions in modified theories of gravity (namely $f(R)$ gravity) has received attentions in recent years [1]. Parallel to black holes there are studies on wormhole solutions in $f(R)$ gravity as well [2]. Upon derivation of the necessary conditions for the existence of black holes in $f(R)$ gravity [3], it will be in order to explore similar conditions for wormholes in the same theory. This amounts to make series expansions for functions around the throat radius ($r=r_0$) of the wormhole. That is, any function $h(r)$ can be expressed as $h(r)=h(r_0)+h'(r_0)(r-r_0)+\mathcal{O}((r-r_0)^2)$. Such expansions are carried out for the Ricci scalar R, $f(R)$, $F=\frac{df}{dR}$ and higher derivatives, as well as the pressure and density of the energy-momentum tensor. Let us remind that even in the absence of external sources the curvature sources of $f(R)$ theory remain intact. The reason for resorting these expansions is entirely physical: near the throat radius of a wormhole designated by $r-r_0=x>0$, ($x^2\ll 1$) an observer doesn't feel anything unusual so that the functions specifying all salient features are expressible in analytic functions. Once these expansions are substituted into Einstein's equations and zeroth, first and higher order terms in $|r-r_0|$ are identified automatic constraints emerge. These will comprise the necessary conditions for the existence of wormhole throats in any viable $f(R)$ theory as a result of such a "*near-throat test*". At the zeroth order the conditions are easily tractable and yield tangible results to say enough about an $f(R)$ whether it admits a throat or not. We must note that the higher order terms, even the first one turns out to yield rather complicated relations in implementing the test. In this study we concentrate on an $f(R)$ theory that admits at least a non-zero third order derivative, i.e. $\frac{d^3f}{dR^3}\neq 0$, while all the rest will be dictated by Einstein's equations. In this study our ansatz metric involves two unknown metric functions to be determined: the redshift function $\Phi(r)$ and the shape function $b(r)$. Upon expansions aforementioned in powers of $(r-r_0)$ there will be severe restrictions on these functions. The simpler case consists of choosing $\Phi(r)$ as constants, however, herein this will not be our strategy. One thing observed is that in the absence of matter sources the weak energy condition (WEC) is violated in the construction of wormholes. To overcome this problem we introduce external energy-momentum and search for the satisfaction of the energy conditions. We find that WEC holds true while the strong energy condition (SEC) becomes valid under more stringent regulations. Integration of $f(R)$ incorporates constants that are related to the cosmological constant and with / without that constant validity of our energy conditions remains still valid. After we determine the redshift and shape functions analytically we embed our geometry into the (r,z,ϕ) sector with the embedding function $z=\pm r_0\cosh^{-1}\left(\frac{r}{r_0}\right)$. The shape of our traversable wormhole is depicted in Fig. 1 for the particular throat radius $r_0=1$.

The paper is organized as follows. In Sec. II we introduce our formalism of expansion around the throat in a generic $f(R)$ theory. Section III extends / generalizes the formalism in the presence of external matter sources. With Conclusion in Section IV we complete the paper.

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II. THE FORMALISM

We start with a general action for $f(R)$ gravity written as

$$S=\frac{1}{2\kappa}\int\sqrt{-g}f\left(R\right)d^{4}x \tag{1}$$

in which $\kappa=8\pi G=1$, and $f\left(R\right)$ is a real arbitrary function of the Ricci scalar R. Obviously, in the presence of physical sources the action is to be supplemented by a matter part S_M. The 4−dimensional standard form of the wormhole line element in spherical symmetry is given by [4]

$$ds^{2}=-e^{2\Phi(r)}dt^{2}+\frac{1}{\left(1-\frac{b(r)}{r}\right)}dr^{2}+r^{2}\left(d\theta^{2}+\sin^{2}\theta d\varphi^{2}\right) \tag{2}$$

where $\Phi\left(r\right)$ and $b\left(r\right)$ are the redshift and shape functions, respectively. The throat of the wormhole is located at $r=r_0$, at which $b\left(r_0\right)=r_0$ and radial coordinate r is larger than r_0. We note that $\Phi\left(r\right)$ and $b\left(r\right)$, in addition to R, $f(R)$ and its higher derivatives should also satisfy constraints to have a traversable wormhole. These conditions for $b(r)$ and $\Phi\left(r\right)$ are: i) $b'\left(r_0\right)<1$, ii) $b\left(r\right)<r$ for $r>r_0$ and iii) $e^{2\Phi(r)}$ must not have any root (horizon) i.e., $\Phi\left(r\right)$ must be finite everywhere.

Variation of the action with respect to the metric yields the field equation

$$FR_{\mu}^{\nu}-\frac{f}{2}\delta_{\mu}^{\nu}-\nabla^{\nu}\nabla_{\mu}F+\delta_{\mu}^{\nu}\Box F=0 \tag{3}$$

in which $\Box=\nabla^{\mu}\nabla_{\mu}=\frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}\partial^{\mu}\right)$ and $\nabla^{\nu}\nabla_{\mu}h=g^{\lambda\nu}\nabla_{\lambda}h_{,\mu}=g^{\lambda\nu}\left(\partial_{\lambda}h_{,\mu}-\Gamma_{\lambda\mu}^{\beta}h_{,\beta}\right)$. The field equations explicitly read as

$$FR_{t}^{t}-\frac{f}{2}+\Box F=\nabla^{t}\nabla_{t}F \tag{4}$$

$$FR_{r}^{r}-\frac{f}{2}+\Box F=\nabla^{r}\nabla_{r}F \tag{5}$$

$$FR_{\theta}^{\theta}-\frac{f}{2}+\Box F=\nabla^{\theta}\nabla_{\theta}F \tag{6}$$

which are independent. Note that the $\varphi\varphi$ equation is identical with $\theta\theta$ equation. By adding the four equations (i.e., tt, rr, $\theta\theta$ and $\varphi\varphi$) we find

$$FR-2f+3\Box F=0 \tag{7}$$

which is the trace of Eq. (3).

Our method to solve these equations is as follows: First, we consider the Ricci scalar as a series about the throat i.e.,

$$R=R_0+R_0'x+\frac{1}{2}R_0''x^2+.... \tag{8}$$

in which $x=r-r_0>0$. Herein and in the sequel a prime stands for the derivative with respect to r and a sub 0 implies that such a quantity is evaluated at the throat. Next, we expand all other functions involved in the field equations i.e.,

$$\Phi = \Phi_0+\Phi_0'x+\frac{1}{2}\Phi_0''x^2+... \tag{9}$$

$$b = b_0+b_0'x+\frac{1}{2}b_0''x^2+... \tag{10}$$

$$f = f_0+f_0'x+\frac{1}{2}f_0''x^2+... \tag{11}$$

$$F = \frac{df}{dR}=F_0+F_0'x+\frac{1}{2}F_0''x^2+... \tag{12}$$

$$E = \frac{d^2f}{dR^2}=E_0+E_0'x+\frac{1}{2}E_0''x^2+... \tag{13}$$

$$H = \frac{d^3f}{dR^3}=H_0+H_0'x+\frac{1}{2}H_0''x^2+... \tag{14}$$

and finally by equating different powers of x we find all coefficients in terms of r_0, R_0, R_0', ... In zeroth order we find

$$b_0 = r_0;\ b_0' = \frac{1}{3}\left(r_0^2 R_0 - 1\right);\ \Phi_0' = -\frac{r_0^2 R_0 + 2}{r_0\left(4 - r_0^2 R_0\right)}; F_0 = \frac{r_0^2 f_0}{r_0^2 R_0 - 2}; \tag{15}$$

$$E_0 = -\frac{2 r_0 f_0}{\left(r_0^2 R_0 - 2\right) R_0'};\quad f_0' = \frac{R_0' r_0^2 f_0}{r_0^2 R_0 - 2}; F_0' = -2\frac{r_0 f_0}{r_0^2 R_0 - 2}; \tag{16}$$

while to first order in x we obtain

$$b_0'' = \frac{1}{3r_0}\left(R_0' r_0^3 + r_0^2 R_0 - 8\right); \Phi_0'' = -\frac{r_0^4 R_0^2 + 8 - 2r_0^2 R_0 + 2r_0^3 R_0'}{r_0^2\left(r_0^2 R_0 - 4\right)^2}; \tag{17}$$

$$H_0 = 2\frac{f_0\left(R_0'' r_0 + R_0'\right)}{R_0'^3\left(r_0^2 R_0 - 2\right)};\ E_0' = 2\frac{f_0\left(R_0'' r_0 + R_0'\right)}{R_0'^2\left(r_0^2 R_0 - 2\right)};\ f_0'' = \frac{r_0 f_0\left(R_0'' r_0 - 2R_0'\right)}{r_0^2 R_0 - 2}. \tag{18}$$

We note that in our solution Φ_0 and f_0 are not specified. As one can see from the line element Φ_0 can be absorbed into the redefinition of time t, so we set it to be zero. On the other hand f_0 is a principal constant which we leave free so that other constants can be expressed in terms of it i.e., $\frac{F_0}{f_0} = \frac{r_0^2}{r_0^2 R_0 - 2}$ and so on. Clearly this is just the zeroth order approximation and it can not be considered for the generic solution.

In order to find the conditions $b(r)$ must satisfy we start with $b'(r_0) < 1$, which implies

$$r_0^2 R_0 < 4. \tag{19}$$

The other condition i.e., $b(r) < r$ for $r > r_0$ is automatically satisfied since

$$\frac{1}{3}\left(r_0^2 R_0 - 1\right) + \frac{1}{2} b_0'' x + ... < 1 \tag{20}$$

and for small x it leads to $r_0^2 R_0 < 4$ which is assumed valid. The last condition which constrains $\Phi(r)$ to be finite takes the form

$$\Phi(r) = -\frac{r_0^2 R_0 + 2}{r_0\left(4 - r_0^2 R_0\right)} x - \frac{r_0^4 R_0^2 + 8 - 2r_0^2 R_0 + 2r_0^3 R_0'}{2r_0^2\left(r_0^2 R_0 - 4\right)^2} x^2 + ... \tag{21}$$

This is guaranteed if x is small enough to keep $\frac{x}{\left(4 - r_0^2 R_0\right)}$ finite together with $r_0^2 R_0 < 4$.

Now let's consider the energy conditions. If we directly write the field equations in the form of standard Einstein equation

$$G_\mu^\nu = \frac{1}{F} T_\mu^\nu + \check{T}_\mu^\nu \tag{22}$$

in which G_μ^ν stands for the Einstein's tensor, with

$$\check{T}_\mu^\nu = \frac{1}{F}\left[\nabla^\nu \nabla_\mu F - \left(\Box F - \frac{1}{2} f + \frac{1}{2} R F\right)\delta_\mu^\nu\right] \tag{23}$$

and T_μ^ν is the external stress energy tensor which in the present case is zero. Next, we consider the energy density ρ and the pressure components produced by the geometry of $f(R)$ gravity in the form [5]

$$\check{T}_\mu^\nu = diag\left[-\rho(r), p_r(r), p_t(r), p_t(r)\right]. \tag{24}$$

After some manipulation we find at the throat (i.e. from the zeroth order Einstein's equations)

$$\rho = -\check{T}_0^0 = \frac{r_0^2 R_0 - 1}{3r_0^2}, \tag{25}$$

$$p_r = \check{T}_1^1 = -\frac{1}{r_0^2} \tag{26}$$

and

$$p_\theta = p_\varphi = \check{T}_2^2 = \frac{1 - r_0^2 R_0}{3r_0^2}. \tag{27}$$

It is observed that imposing $1 < r_0^2 R_0 < 4$, makes ρ positive but $p_\theta = p_\varphi$ remain negative and in any case p_r is negative. Note that it may be this negative pressure that protects the wormhole against collapse. From the summation of (25) and (26) it can be seen easily that the weak energy condition (WEC) which says that $\rho + p_i \geqslant 0$ and $\rho \geqslant 0$, is violated. In the next section we may avert this situation by adding external sources.

A. Wormhole supported by a matter source

In this section we consider a matter source which provides an energy-momentum tensor of the form

$$T_\mu^\nu = diag\left[-\rho, p, q, q\right] \tag{28}$$

in which ρ, p and q are arbitrary functions of r. The field equations, read now

$$G_\mu^\nu = \frac{1}{F} T_\mu^\nu + \check{T}_\mu^\nu. \tag{29}$$

The same method as we used in the previous section, i.e., expansion of quantities about the throat, including ρ, p and q

$$\rho = \rho_0 + \rho_0' x + ... \tag{30}$$

$$p = p_0 + p_0' x + ... \tag{31}$$

$$q = q_0 + q_0' x + ... \tag{32}$$

would lead to the following results in the zeroth order:

$$\rho_0 = \frac{1}{2} f_0 + \frac{F_0' \left(b_0' - 1\right)}{2r_0} - \frac{\left(b_0' - 1\right) \Phi_0' F_0}{2r_0}, \tag{33}$$

$$p_0 = -\frac{1}{2} f_0 + \frac{F_0 \left(b_0' - 1\right) \left(\Phi_0' r_0 + 2\right)}{2r_0^2} \tag{34}$$

and

$$q_0 = -\frac{1}{2} f_0 + \frac{F_0 \left(b_0' + 1\right)}{2r_0^2} - \frac{F_0' \left(b_0' - 1\right)}{2r_0}. \tag{35}$$

Next, we try to apply WEC which implies $\rho_0 \geq 0$ and $\rho_0 + p_{i0} \geq 0$. One finds the following conditions which should be satisfied simultaneously:

$$\rho_0 \geq 0 \rightarrow f_0 - \frac{F_0' \left(1 - b_0'\right)}{r_0} + \frac{\left(1 - b_0'\right) \Phi_0' F_0}{r_0} \geq 0, \tag{36}$$

$$\rho_0 + p_0 \geq 0 \rightarrow r_0 F_0' + 2F_0 \leq 0. \tag{37}$$

and

$$\rho_0 + q_0 \geq 0 \rightarrow F_0 \left[1 + r_0 \Phi_0' \frac{1 - b_0'}{1 + b_0'}\right] \geq 0 \tag{38}$$

in which we used the choice that $b_0' - 1 < 0$. Next, one may add to the foregoing conditions $\rho_0 + p_0 + 2q_0 \geq 0$ which implies the strong energy conditions (SEC) and gives

$$\frac{2b_0' F_0}{r_0^2} + \frac{F_0' \left(1 - b_0'\right)}{2r_0} - f_0 \geq 0. \tag{39}$$

In the case of $\Phi' = 0$ [5] one obtains

$$WEC \rightarrow f_0 \geq \frac{F_0' \left(1 - b_0'\right)}{r_0};\ \frac{F_0'}{F_0} \leq -\frac{2}{r_0};\ F_0 \geq 0 \text{ (and consequently } F_0' \leq 0\text{)}\ ; \tag{40}$$

and

$$SEC \rightarrow WEC \text{ and } 0 \leq \frac{f_0}{F_0} - \frac{F_0' \left(1 - b_0'\right)}{F_0 r_0} \leq \frac{2b_0'}{r_0^2} - \frac{F_0' \left(1 - b_0'\right)}{2F_0 r_0}. \tag{41}$$

We note that although these conditions are very complicated in principle one can choose the proper parameters to satisfy them. In the next section we give an exact solution to the problem and with a particular example we shall justify our prediction.

III. GENERALIZATION

In this section we study the field equations not at the throat but at any $r \geqslant r_0$. This is also a generalization of Ref. [5] in which Φ =constant. The field equations are given in Eq.s (4)-(7) and the line element is considered as (2) while the energy momentum is given by (28). A detailed calculation would imply the following relations for the components of energy momentum:

$$\rho = \frac{f}{2} + \frac{2r\left(\Phi'' + \Phi'^2\right)(r-b) + \left[-3b + (4-b')\,r\right]\Phi'}{2r^2}F + \frac{(b'-4)\,r + 3b}{2r^2}F' - \left(1 - \frac{b}{r}\right)F'', \tag{42}$$

$$p = -\frac{f}{2} + \frac{2r^2\,(b-r)\left(\Phi'^2 + \Phi''\right) + (r\Phi' + 2)\,(b'r - b)}{2r^3}F + \frac{(r-b)\,(r\Phi' + 2)}{r^2}F', \tag{43}$$

$$q = -\frac{f}{2} + \frac{2r\,(b-r)\,\Phi' + rb' + b}{2r^3}F - \frac{\left[-2r\,(r-b)\,\Phi' + (b'-2)\,r + b\right]}{2r^2}F' + \left(1 - \frac{b}{r}\right)F''. \tag{44}$$

The case of Ref. [5] is easily observed if one sets Φ =constant and therefore

$$\rho = \frac{f}{2} + \frac{(b'-4)\,r + 3b}{2r^2}F' - \left(1 - \frac{b}{r}\right)F'', \tag{45}$$

$$p = -\frac{f}{2} + \frac{(b'r - b)}{r^3}F + \frac{2\,(r-b)}{r^2}F', \tag{46}$$

$$q = -\frac{f}{2} + \frac{rb' + b}{2r^3}F - \frac{(b'-2)\,r + b}{2r^2}F' + \left(1 - \frac{b}{r}\right)F''. \tag{47}$$

The other limit can be checked for $r = r_0$ ($b = b_0$) and the results found in (33)-(35) are recovered.

Next, let's consider an isotropic velocity distribution which implies $p = q$ together with the ansaetze $b(r) = r_0^2/r$ and $\Phi\,(r) = 2\ln\left(\frac{r}{r_0}\right)$. These in turn lead to

$$r^2\left(r^2 - r_0^2\right)F'' + r\left(2r_0^2 - r^2\right)F' + 4r_0^2 F = 0. \tag{48}$$

This equation admits two independent solutions and to keep our calculation analytic, we choose the simpler one given by

$$F = C_1\sqrt{1 - \frac{r_0^2}{r^2}}, \tag{49}$$

in which C_1 is an integration constant. Same ansatz in the Ricci scalar gives

$$R = \frac{6\left(r_0^2 - 2r^2\right)}{r^4} \tag{50}$$

which implies (through $F = \frac{df}{dR} = \frac{f'}{R'}$)

$$f = \frac{24C_1}{5r_0^2}\left(1 - \frac{r_0^2}{r^2}\right)^{5/2} + C_2 \tag{51}$$

for the integration constant C_2 that is related with the cosmological constant. Finally one obtains

$$f\,(R) = \frac{24C_1}{5r_0^2}\left(1 + \frac{\left(\frac{r_0^2 R}{6}\right)}{1 + \sqrt{1 + \left(\frac{r_0^2 R}{6}\right)}}\right)^{5/2} + C_2. \tag{52}$$

We note that the Ricci scalar R satisfies $\frac{-6}{r_0^2} < R < 0$. This causes no restriction on the above expression and is well defined everywhere for $r \geq r_0$. Our next step is to find the stress-energy components which are as follow:

$$\rho = \frac{\left(24r^6 - 12r^4r_0^2 - 18r^2r_0^4 + 6r_0^6\right)C_1 + 5r^5r_0^2C_2\sqrt{r^2 - r_0^2}}{10r_0^2r^5\sqrt{r^2 - r_0^2}}, \tag{53}$$

$$p = q = -\frac{\left(24r^6 - 52r^4r_0^2 + 32r^2r_0^4 - 4r_0^6\right)C_1 + 5r^5r_0^2C_2\sqrt{r^2 - r_0^2}}{10r_0^2r^5\sqrt{r^2 - r_0^2}}. \tag{54}$$

Furthermore the WEC implies that $\rho \geq 0$ and $\rho + p \geq 0$, which in closed form read

$$\rho \geq 0 \rightarrow \left(24r^6 - 12r^4r_0^2 - 18r^2r_0^4 + 6r_0^6\right)C_1 + 5r^5r_0^2C_2\sqrt{r^2 - r_0^2} \geq 0 \tag{55}$$

and

$$\rho + p \geq 0 \rightarrow \frac{C_1\left(4r^4 - 5r^2r_0^2 + r_0^4\right)}{r^5\sqrt{r^2 - r_0^2}} \geq 0. \tag{56}$$

One can see that both conditions are satisfied if both C_1 and C_2 remain positive. In addition to WEC it is remarkable that the strong energy condition (SEC) i.e. $\rho + p \geq 0$ and $\rho + 3p \geq 0$, is also satisfied under the following condition:

$$\frac{C_2}{C_1} \leq \frac{3}{5}\frac{3 - 16\xi^2 + 8\xi^4}{r_0^2\xi^5}\sqrt{\xi^2 - 1}. \tag{57}$$

Here C_1 and C_2 both are positive as chosen above for WEC and $\xi\left(= \frac{r}{r_0}\right)$ is a positive parameter such that $1 < \xi < \frac{\sqrt{4+\sqrt{10}}}{2}$. It should be added that even for the case of $C_2 = 0$, the SEC holds true in the particular range of ξ. Let us add that an expansion of $f(R)$ in powers of R yields

$$f(R) \cong \frac{24C_1}{5r_0^2} + C_2 + C_1R + \mathcal{O}(R^2) \tag{58}$$

which gives the exact combination of integration constants (C_1, C_2) that can be identified as the cosmological constant. The weak-curvature expansion (58) suggests that $C_1 \neq 0$, is the crucial constant in order to attain Einstein-Hilbert term, but C_2 can be disposed.

Finally we look at the wormhole's line element

$$ds^2 = -\left(\frac{r}{r_0}\right)^4 dt^2 + \frac{dr^2}{1 - \frac{r_0^2}{r^2}} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{59}$$

and check the conditions which $b(r) = \frac{r_0^2}{r}$ and $\Phi(r) = 2\ln\left(\frac{r}{r_0}\right)$ must satisfy. The first condition which states that $\frac{b - b'r}{b^2} > 0$ is satisfied since

$$\frac{b - b'r}{b^2} = \frac{2r}{r_0^2} > 0. \tag{60}$$

The second condition implies that $b'(r_0) < 1$. This in turn reads $-r_0 < 1$ and is obvious. Finally $1 - \frac{b}{r} > 0$ means $1 - \frac{r_0^2}{r^2} > 0$ and is also satisfied for $r > r_0$. The condition on $\Phi(r)$ implies that it must be finite everywhere without root for $r > r_0$ which is trivially satisfied (except for $r \rightarrow \infty$). To complete our example we give the embedded surface as $z(r) = r_0\cosh^{-1}\left(\frac{r}{r_0}\right)$. This is shown in Fig. 1 for $r_0 = 1$.

IV. CONCLUSION

One of the most challenging problems in wormhole physics is to find an acceptable energy-momentum that will provide the outward push against gravitational collapse. In Einstein's general relativity, i.e. $f(R) = R$, this has

not been possible. Now, with the advent of modified theories, namely the $f(R)$ gravity, this may turn into reality. Our approach to the problem of wormholes is to employ necessary existence conditions analogous to black holes. The existence problem of horizon in a black hole plays the similar role for throat radius in a wormhole. Expansion of metric functions around the throat and substitutions into Einstein's equations derive the necessary conditions. From these conditions, at the lowest order, we determine the metric functions that yield a traversable wormhole. In this process we have assumed that $f(R)$ admits non-zero derivatives at least to third order, i.e. $\frac{d^3 f}{dR^3} \neq 0$. With the introduction of external matter sources into $f(R)$ we show that the weak energy condition (WEC) (at least) is satisfied in the construction of a wormhole in $f(R)$ gravity. The $f(R)$ function is explicitly determined (Eq. (52)) in which the Ricci scalar satisfies $\frac{-6}{r_0^2} < R < 0$. Our results apply also to the case with / without a cosmological constant since the latter arises automatically as a combination of integration constants. In conclusion, in a generic class of $f(R)$ theories satisfying the necessary conditions for existence of traversable wormhole, solutions can be supported by physical stress-energy tensor. The embedding diagram for the obtained wormhole is shown in Fig. 1.

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Figure Caption:

Fig. 1: The embedding surface (i.e. $dr^2 + dz^2 = \frac{dr^2}{1-\left(\frac{r_0}{r}\right)^2}$ and $\theta = \frac{\pi}{2}$) for $b(r) = \frac{r_0^2}{r}$ and $\pm z(r) = r_0 \cosh^{-1}\left(\frac{r}{r_0}\right)$ for $r_0 = 1$.